Transposition Invariance and Parsimonious Relation of Z Sets

Introduction

This article presents some new research into Z sets, that is, set classes without a unique interval class vector. Some novel results on self-transposing Z sets and the parsimonious relation of Z sets are given. The theorems are contextualised with respect to work of Dan Tudor-Vuza (Tudor-Vuza 1993), Stephen Soderberg (Soderberg 1995) and other music theorists who have tried to breach the defenses of Z sets as mathematical entities. The seemingly innocuous same interval vector property proves rather more intransigent under the microscope.

The relevance of Z sets to musical analysis is undisputed (see for instance the classic reference of Forte 1973). They provide an attractive resource for composers in allowing a match of intervallic content between distinct set classes, and hence a mechanism of ambiguity and modulation.

Where practical, proofs are kept as informal as possible. The reader may skip any mathematics as it appears in the text, but will otherwise require familiarity with group actions.

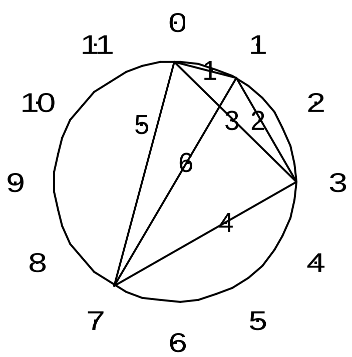
Before racing into results on Z sets, it would be wise to prepare our ground.

Notation

Unfortunately, notations vary from paper to paper. (Fripertinger 1999) gives a fully rigorous exposition of the algebraic combinatorics of music theory. To avoid an excess of notation, matters are simplified from that worthy text. The set of pitch classes 0 to n-1 is represented by the mathematical space $\mathbf{Z}_n$, a ring under modulo arithmetic addition and multiplication. Equivalence classes can be established on the power set of $\mathbf{Z}_n$ under the action of the dihedral group generated by the well known operations of transposition T and inversion I. These equivalence classes are called (<T, I>) set classes by music theorists and (Morris 87) provides a good introduction to these matters. We can speak of set classes without qualification since we always deal herein with set classes with respect to transposition and inversion. The set of all set classes for $\mathbf{Z}_n$ will be referenced in a relaxed way as if set classes 'belong' to $\mathbf{Z}_n$. The reader should understand from the context what particular construction on $\mathbf{Z}_n$ is under investigation, and this will allow us to avoid introducing an extra layer of notation.

We denote the set class of a representative pitch class set A, /A/, following (Lewin 1987). After Lewin again, if card A = M, then /A/ is a M-class.

This paper deals only with standard interval class vectors as present in Forte's famous list (Forte 1973, Appendix 1). The definition of the numerical interval between two pitch classes is the 'geodesic' definition. It might be helpful to think of the shortest possible distance around the outside of a Krenek diagram, as demonstrated in figure 1.



4-Z29

Figure 1 Krenek diagram showing the intervals in 4-Z29

Definition Given a, b in $\mathbf{Z}_n$,

int(a, b) = min (b-a, n+a-b) where $n > b \geq a \geq 0$

The set of possible values of the int function over $\mathbf{Z}_n$ is { 0, …, (n-1)/2 }for odd n, or { 0, …, n/2 }for even. These values are the representatives of what Forte calls interval classes. The interval classes contain those integers of $\mathbf{Z}_n$ in the inverse image under int. In $\mathbf{Z}_{12}$ these would be

/0/ = {0}, /1/= {1,11}, /2/= {2,10}, /3/= {3,9}, /4/= {4,8}, /5/= {5,7}, /6/ = {6} where /r/ again denotes an equivalence class with representative r. In the following, interval classes will be treated as synonymous with the (common) value of the function int across that class.

The interval vector for pitch class set A can be constructed by considering all possible pairs of pitch classes in A[1]. Taking the interval value for each pair, the interval vector will sort the data by holding the multiplicity of occurrence of each possible interval. Since intervals are invariant under transposition and inversion, the interval vector for a set class /A/ is equal to that for any representative A.

If two distinct set classes have the same interval vector they are called Z partners. The term Z pair must be avoided in general since Z triples and Z collections with larger numbers of solutions can occur. Lewin was the first to find such examples; Soderberg reveals more in his cited paper. Individually a partner is a Z set, or Z chord.

We deal with orbits of the transposition operation T_j on $\mathbf{Z}_n$. Where j divides n these orbits were termed Φ-sets in Stephen Soderberg's 1995 paper. For example, the orbits of T_2 in $\mathbf{Z}_{12}$ are the complementary whole tone sets {0, 2, 4, 6, 8, 10} and {1, 3, 5, 7, 9, 11} and Soderberg would write them Φ(12, 0, 2) and Φ(12, 1, 2) respectively (section 3, page 82, Soderberg 1995). We shall use and extend this notation for dealing with a collection of orbits of T_j. For any j, n, and collection of representatives A_I,

$$\bigcup_{k \in A_I} \Phi(n, k, j)$$

is the union of the orbits of T_j in $\mathbf{Z}_n$ with index set of representatives A_I. We do not require that j | n, but will often deal with the case where (j, n) >1, that is, the greatest common divisor of j and n is greater than 1. If (j ,n) = 1, j and n are coprime. Note that the size of the orbit of T_j in $\mathbf{Z}_n$ is given by n/(j, n). A union of such complete orbits will have cardinality some integer multiple of this number.

It is helpful to review two fundamental theorems for Z sets. The first asserts that Z partners must have the same M-class. This follows by a cardinality argument, noting that the number of ways of choosing distinct pairs from a set of M elements (M ≥ 2) is M!/(M-2)!2!= M.(M-1)/2, and the value of this as M goes from 2 to n in $\mathbf{Z}_n$ is always distinct. The second is the well known Babbitt's theorem, applying only in $\mathbf{Z}_n$ for even n. This states that the complement of a (n/2)-set has the same interval vector. So if the complement is not the set class itself, it must be a Z partner. Proofs of this result occur in (Soderberg 1995), (Lewin 1987) and a wonderfully concise proof based on group tables is in (Wilcox 1983).

We are now in a position to tackle some original results on Z sets.

Z Sets and the Size of the Entries in an Interval Vector

Can we obtain any information about uniqueness of interval vectors from numerical entries appearing in the vector? It turns out that this is not the case excepting in a very specific circumstance.

Proposition In space $\mathbf{Z}_n$ let the entry of interval j in the interval vector of set class /A/ be num(j), where (j , n) = 1 (j and n are coprime). Then if num(j) $\geq$ (card A -1), /A/ has no Z partner.

Proof By consideration of the orbits of $\mathbf{Z}_n$ under T_j. Since j and n have no common divisor greater than 1, there is only one orbit. Write this orbit

$O(T_j)= \{0, 0+j, 0+2j \bmod n, 0+3j \bmod n, \ldots, 0 + (n-1)j \bmod n\}$.

Any interval of size j can be traced to two consecutive elements in the orbit.
If $O(T_j)$ is of size 2, then the space is the trivial space $\mathbf{Z}_2$, and we can disregard that case.

There are at most n possible intervals of size j available (we must include the looparound from (n-1)j mod n back to 0). In order to obtain k intervals of size j from k elements, we would have to take all n pitch classes, since we need the looparound. But this tells us that /A/= /{0,…,n-1}/, which is always the singular n-class, and hence cannot have a Z partner. To obtain k-1 intervals of size j from k elements, we must take some consecutive chain subset from the orbit $O(T_j)$, as {Lj mod n, …, (L+k-1)j mod n} for some integer L. (If this is not the case, then we have at least two separate chains. Each chain of elements from the orbit generation can only provide p intervals of size j for p+1 elements in the chain. So two chains can at best provide p-2 intervals from p selected elements). Up to transposition and inversion, this specifies k elements exactly. Since k= card A for our case, we have determined a unique set class from a given interval vector. So /A/ has no Z partners.

Counterexamples can be given to the above proposition, when (j, n) > 1. For in such a circumstance, there is more than one orbit of T_j acting on $\mathbf{Z}_n$. By choosing different combinations of orbits, we can produce the same interval vector for different set classes.

A computer search reveals the first example of entries of size card A -1 in an interval vector in $\mathbf{Z}_{20}$.

A= { 0, 1, 2, 5, 6, 7, 10, 12, 15, 17 }

B= { 0, 1, 3, 5, 6, 8, 10, 11, 15, 16 }

IV(/A/) = [4, 4, 4, 4, 9, 4, 4, 4, 4, 4] = IV(/B/)

(card A = card B = 10)

Note that A could be written as two complete orbits of T_5 plus the incomplete {1,6}, and B the two full orbits with {3,8} left over.

The first example for an entry of size card A in an interval vector appears in $\mathbf{Z}_{24}$. It is the unique example in the quartertone space. We may use Soderberg's notation to write the Z pair succinctly as a union of orbits of T_8. Let the index sets of representatives be A_I = { 0, 1, 2, 5 } and B_I = { 0, 1, 3, 4 }. Then the Z partners A and B are[2]:

A = $\bigcup_{k \in A_I} \Phi(24, k, 8)$ and B = $\bigcup_{k \in B_I} \Phi(24, k, 8)$

IV(/A/) = [6, 3, 6, 6, 6, 3, 6, 12, 6, 3, 6, 3] = IV(/B/)

This example shows us directly that Z sets can be self-transposing. The necessary and sufficient condition for a self transposing set class /A/ is that A be constructed as a union of complete T_j orbits. Then under T_j, A is transposition invariant. In the situation above, the interval vector of /A/ had an entry (for interval class j) of size card A; which must correspond to A being a union of complete Tj orbits.

Now, note that in $\mathbf{Z}_{16}$

A = $\bigcup_{k \in A_I} \Phi(16, k, 8)$ and B = $\bigcup_{k \in B_I} \Phi(16, k, 8)$

IV(/A/) = [4, 2, 4, 4, 4, 2, 4, 4] = IV(/B/)

for the same A_I and B_I introduced above. Suddenly, we have self transposing Z partners A and B, with an entry for interval class 8 in the interval vector of 4. To explain this, remember that complete orbits of size 2 only provide one interval class per orbit.

We are close to a classification of all self transposing Z sets. As a second vital clue, it is not a coincidence that our A_I and B_I are actually Z sets in $\mathbf{Z}_8$. In fact, we can take the observations above as the basis of a proposition about self transposing Z sets.

Proposition Given M class /A/, suppose A be a union of orbits of T_j in $\mathbf{Z}_n$ with (j, n) >1. Write the index set of representatives for these orbits

$A_I = \{ a_1, \ldots, a_p \}$ where all a_i satisfy $0 \leq a_i < j$, and $p(n / (j, n)) = M$.

Similarly construct some B and B_I.

A = $\bigcup_{k \in A_I} \Phi(n, k, j)$ and B = $\bigcup_{k \in B_I} \Phi(n, k, j)$

Then[3] A is in the Z relation with B in $\mathbf{Z}_n$ iff A_I is in the Z relation with B_I in $\mathbf{Z}_{(j, n)}$.

Note that when (j, n)=1 this theorem tells us nothing helpful; namely that A and B are Z partners in $\mathbf{Z}_n$ iff they are Z partners in $\mathbf{Z}_n$! But in this case, A=B=$\mathbf{Z}_n$ by the conditions of the proposition, since the orbit of Tj in $\mathbf{Z}_n$ is $\mathbf{Z}_n$.

Proof We must compare the idea of an interval in $\mathbf{Z}_{(j, n)}$ with intervals in $\mathbf{Z}_n$.

Suppose $i_1 \neq i_2$ are in the same interval class in $\mathbf{Z}_{(j, n)}$. Then

$$i_1 + i_2 = (j, n) \qquad (1)$$

Write

$i_1 + q\,(n, j) + n - i_1 - q(n, j) = n$ (2)

for any $q \in \{ 0, \ldots, n/(j, n) -1 \}$

The equation represents that $i_1 + q(j, n)$ and $n - i_1 - q(j, n)$ are in the same interval class in $\mathbf{Z}_n$. We substitute into (2) as follows:

$i_1 + q(j, n) + n + i_2 - (j, n) - q(j, n) = n$ using (1)

$i_1 + q(j, n) + i_2 + (n/(j, n) - q -1)(j, n) = n$

So $i_1 + q(j,n)$ is in the same interval class as $i_2 + q'j$ where $q'= (n/(j, n) -q -1)$.
As q takes on successive integers from 0 to $n/(j, n) - 1$, q' covers $n/(j, n) - 1$ to 0 respectively.

The set of interval classes in $\mathbf{Z}_n$ generated in this manner from a particular interval class in $\mathbf{Z}_{(j, n)}$ are the same. So given orbits of T_j in $\mathbf{Z}_n$, the set of interval classes arising between them is the same iff the representatives of those orbits (taken of course in the range $0 \leq a < j$) are in the same interval class in $\mathbf{Z}_{(j, n)}$. This proves the proposition.

The cardinalities involved imply that in order to find self-transposing Z sets we can utilise well known Z sets. The first occurring Z sets are in $\mathbf{Z}_8$ as given above. The smallest non trivial orbit is of cardinality two. The first example of self transposing Z sets occurs in $\mathbf{Z}_{16}$. It is a unique example because there is only one pair of Z sets in $\mathbf{Z}_8$. The proposition is useful for construction; it tells us when the next examples can occur. This is in $\mathbf{Z}_{20}$, because we can find Z sets in $\mathbf{Z}_{10}$ and use the size 2 orbits of T_{10} in $\mathbf{Z}_{20}$. There should be three such examples since there are three distinct pairs of Z partners in $\mathbf{Z}_{10}$.

The proposition can be simplified in its application, as the orbits of T_j in $\mathbf{Z}_n$ are the same as the orbits of $T_{(j, n)}$ in $\mathbf{Z}_n$. We need only worry about j such that $(j, n)=j$, since any other situation will simply provide duplicates of the Z sets we discover in the simpler case.

The reader may be interested to know that the proposition would allow the construction of an infinite set of self-transposing Z sets across all $\mathbf{Z}_n$:

Proposition Self-transposing Z sets occur in $\mathbf{Z}_n$ for all n $\geq$ 28 where n can be factored as pq, p $\geq$ 14 and q $\geq$ 2.

Proof Use the special first INTNOMOD Z set pair from (Collins 2000). They have the property of revealing representatives for Z set classes in $\mathbf{Z}_n$ for all n>=14.

As a final example, we may construct self transposing Z sets in $\mathbf{Z}_{99}$ using the INTNOMOD set pair in $\mathbf{Z}_{33}$ and thus orbits of size 3:

$A_I = \{ 0, 1, 4, 5, 6, 7, 8, 9, 11 \}$ $B_I = \{ 0, 1, 2, 3, 4, 6, 7, 8, 11 \}$

A = $\bigcup_{k \in A_I} \Phi(99, k, 33)$ and B = $\bigcup_{k \in B_I} \Phi(99, k, 33)$

We can spare ourselves the 49 entry interval vector. A computer quickly confirms the validity of the example (even if the power set of $\mathbf{Z}_{99}$ is too big to exhaustively search). Note that /A/ and /B/ are distinct set classes since B involves chromatic runs of length 5 whilst A has ones of length 6.

It is left as an open research problem to consider more than one interval vector entry at once. There are Z sets /A/, /B/ in $\mathbf{Z}_{12}$ with two entries of size card A - 2 in their interval vector, but can we ever have two entries of size card A -1 or above in an interval vector?

Z Sets and Parsimonious Operations

Parsimonious transformation is an extremely current research stream in music theory. We cite the special edition of The Journal of Music theory devoted to work on neo-Riemannian theory, in particular the article on parsimonious graphs (Douthett and Steinbach 1998) from which the notation presented here is lifted.

Definition Pcsets A and B are in the $P_{m,n}$ relation (A $P_{m,n}$ B) iff there exists a permutation σ mapping A to B such that σ is a product of distinct involutions:

$\sigma = (p_1\ p_2) \ldots (p_{2(m+n)-1}\ p_{2(m+n)})$ all p_i distinct and increasing
where the multiplicity of these is such that
int $(p_i, p_i+1) = 1$ for m involutions, and int $(p_i, p_i+1) = 2$ for n

Generalisations of the $P_{m,n}$ relation would allow permutations of more than 2 semitones distance, but we have enough to express the theorem I wish to introduce here:

Proposition If two pcsets A, B in $\mathbf{Z}_n$ satisfy A $P_{1,0}$ B and A has the same interval vector as B, then /A/= /B/.
Proof Without loss of generality, consider only the case where the parsimonious shift is from pitch class 0 to 1. A contains 0 and not 1, B contains 1 and not 0. This is possible since interval content is invariant under transposition and inversion, and A may be freely relabelled such that the shift is leftwards from 0 to 1.

Write
$A = \{0, a_1, \ldots, a_M, a_{M+1}, \ldots, a_{M+N}\}$
$B = \{1, a_1, \ldots, a_M, a_{M+1}, \ldots, a_{M+N}\}$

$A_M= \{a_1, \ldots, a_M \}$, $A_N= \{ a_{M+1}, \ldots, a_{M+N} \}$

$A= A_M \cup A_N \cup \{0\}$
$B= A_M \cup A_N \cup \{1\}$

The a_is are increasing for all i,
where $1< a_i \leq n/2$ for $1 \leq i \leq M$
and $n/2 < a_i < n$ for $M+1 \leq i \leq M+N$.

Under the assumptions that A $P_{1,0}$ B and the interval vector of A equals that of B it will be shown that M = N, and $B= T_1I(A)$ $(B= I_1A)$. This equation is equivalent to the symmetricity of A and B about an axis through 1/2 and n/2+1/2, and shows immediately that /A/= /B/.

The elements a_i of A and B have been split up in such a way that the change of interval class after $P_{1,0}$ can be easily handled. The only intervals that may change under the transformation are those in A written int(0, a_i) becoming int(1, a_i) for B's interval vector. We disregard all other intervals from our consideration.

Note that

int(0, a_i) = a_i and int(1, a_i) = a_i-1 for $1 \leq i \leq M$ (1)

int(0, a_i) = n-a_i and int(1, a_i) = n-a_i +1 for $M+1 \leq i \leq M+N$ (2)

Because of the choice of the a_i, there is no need to worry about the odder aspects of interval classes (in $\mathbf{Z}_{12}$, ic 6 +1 = ic 5 et al). The reader will observe that the operator $P_{1,0}$ shifts the interval classes by one step, and that the shift is in contrary direction for case (1) and (2).

Let I be the greatest interval class occurring in A. This interval must be formed as either int(0, a_{M+1}) or int(0, a_M). If the former alone, then under $P_{1,0}$ this interval increases to n- a_{M+1}+ 1, which does not occur for any int(0, a_i) in A, contradicting the hypothesis that A and B have equal interval vectors. There is one special case for n odd, in which a_{M+1} is on the axis of symmetry through 1/2 and n/2+1/2 and int(0, a_{M+1})= int (1, a_{M+1}), but this case fits the result desired, and we proceed iteratively as below knowing it cannot recur. Otherwise we must have int(0, a_M) as the single largest interval in A of that form. Under $P_{1,0}$, there must be some int(1, a_j) to match it. It is quickly seen that only int(1, a_{M+1}) could be the interval. We have shown that

int(1,a_{M+1}) = n- a_{M+1} +1 = a_M = int(0, a_M) and

int(0,a_{M+1}) = n- a_M+1 = a_M-1= int(1, a_M)

Now proceed iteratively, considering the remaining greatest interval class unaccounted for each time. It will be seen that we pair off a_i such that

int(1, $a_{M+N+1-i}$) = n- $a_{M+N+1-i}$ +1 = a_i = int (0, a_i) $1 \leq i \leq M$ (3)

$\text{int}(0, a_{M+N+1-i}) = n - a_{M+N+1-i} = a_i - 1 = \text{int}(1, a_i)$

and that M = N, else a contradiction must occur as above. (3) says that as pitch classes modulo n

$a_{M+N+1-i} = -a_i + 1 \qquad 1 \le i \le M$

which is really the assertion B= $T_1I(A)$ as desired.

Corollary Z sets cannot be related by a single $P_{1,0}$ shift between representatives.

To assist in a rough understanding of the proof, figure 2 shows the $P_{1,0}$ transformation in the specific case of the set { 0, 3, 6, 9 }in $\mathbf{Z}_{11}$, where the interval vector is preserved. Only those intervals affected by the shift are marked. The axis of inversion is demonstrated through 1/2 and 1/2+11/2. The reader can easily convince themselves that if the pitch class set was { 0, 3, 6, 8 }then the interval vector could not be preserved under $P_{1,0}$.

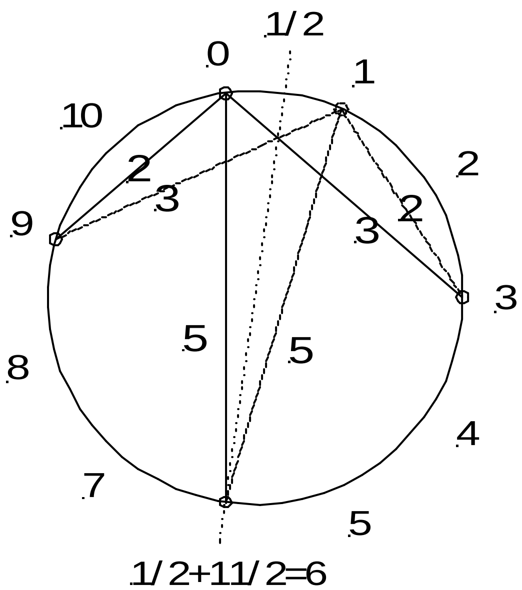


Figure 2 A specific case of preservation of interval vector under $P_{1,0}$

It doesn't take long to discover that for their standard representatives 4-Z15 $P_{2,0}$ 4-Z29 and that 5-Z18 $P_{1,1}$ 5-Z38. The reader might wonder whether interval vectors are preserved under more generally defined P(0,…,0,1,0,…,0) transformations iff /A/ =

/B/. Then note that { 0, 1, 3, 7 }under permutation (0,6) is { 1, 3, 6, 7 }which under T_7I = { 0, 1, 4, 6 }. I might speculate that if a permutation causes a swap of elements p_i and p_j of a pitch class set A such that there exists a further element p_k satisfying $p_i<p_k<p_j$ then no proof is possible. I leave it as an open problem whether a theorem could be proven if this is not the case. Seeking an axis of symmetry for A about q/2 and p/2+ q/2 where q is the step size of the singular involution in the permutation, will familiarise the reader with how the 'choice' of one of two elements of an interval class interferes with proofs!

Towards a Z Set Property List?

To put these new propositions in context, we can try to bring together a current state of knowledge on Z sets.

There's no need to go outside $\mathbf{Z}_{12}$ to find examples of a self inverse Z set. 5-Z12 = /{ 0, 1, 3, 5, 6 }/ has difference set [[1,2,2,1,6]] (notation as Soderberg 1995) which is obviously unaffected by inversion.

Referring to (Clough et al 1997) for the concept of spectrum, further note that the <1> (spectrum of diatonic length 1) is not unique for Z sets. 6-Z17 and 6-Z43 share <1>= {1,1,1,2,3,4} and are not Z partners with each other. It is uncertain whether a proposition could be constructed of general applicability to Z sets based on <2>, <3> etc.

If the reader is discouraged by this point from attempts to prove general assertions about Z sets, they may find table 1 a comfort; the Z sets increasingly saturate the larger pitch class spaces. (This table was created using a computer search with the author's custom C++ program[4]).

The reader who perceives some discrepancy in the table due to an odd count of Z sets is forgetting the existence of Z triples and the like.

Table 1 Frequencies of Z sets for $\mathbf{Z}_n$ 8 £ n £ 24

n	interval classes	#set classes	#Z set classes	# Z from Generalised Hexachord Theorem	proportion (proportion from GHT)
8	4	30	2	2	6.6% (6.6%)
9	4	46	0		0%
10	5	78	6	6	7.69% (7.69%)
11	5	126	0		0%
12	6	224	46	30	20.5% (13.39%)
13	6	380	12		3.16%
14	7	687	144	96	20.96% (13.97%)
15	7	1224	160		13.07%
16	8	2250	728	366	32.36% (16.27%)
17	8	4112	368		8.95%
18	9	7685	2766	1258	35.99% (16.37%)
19	9	14310	1296		9.06%
20	10	27012	10403	4481	38.5% (16.59%)
21	10	50964	9268		18.19%
22	11	96909	32085	15605	33.11% (16.10%)
23	11	184410	15708		8.52%
24	12	352698	162974	55838	46.21% (15.83%)

There are always more Z sets for the even n pitch class spaces because of Babbitt's theorem. The proportions of set classes that are Z sets is surely dependent on the prime decomposition of n, so the proportions fall off for n=19 and n=23. To what degree can Z sets saturate the higher order spaces? The Z sets could never entirely saturate a space since there is always a unique n-class for $\mathbf{Z}_n$. Yet given that 46 percent of set classes in $\mathbf{Z}_{24}$ are Z sets, around half of chord types selected by a quartertonal composer would have Z partners available! With the degree of saturation exhibited, it is hard to predict any of the simpler properties of set classes studied by music theorists providing general propositions.

Let us begin to round up what we know, and what headway other theorists have managed to make.

In terms of existence, (Collins 2000) proves the existence of Z sets across all $\mathbf{Z}_n$ for $n \geq 12$, and the results for the smaller cardinalities are already known by computer search.

Steven Soderberg's paper on dual inversions (Soderberg 1995) introduces the idea of Q-grids, which may assist the discovery of Z sets in some specific cases. Whilst his work may potentially be generalised to unions of more than two orbits of a T_j under tighter conditions, there is currently no proof that every possible Z set must result from some Q grid. Soderberg's work does not guarantee any surefire procedure for constructing Z sets alone, but it does provide a tantalising glimpse of conditions sufficient to create them. It is possible that a general Z set theorem might follow from this work, but a temporary conclusion is that Soderberg has discovered a particular breed of Z sets, call them 'Soderberg Z chords' if you will.

The full problem of Z sets can be set in the context of the algebraic Fourier transform. Lewin was the first to recognise this (Lewin 1987) though the most advanced exposition of this problem is in (Tudor Vuza 1993). Dan Tudor Vuza uses algebraic fourier methods to solve a more restricted case particular to his canon spaces. For the general problem of Z sets the task seems almost impossible without any restriction on the characteristic functions involved. It is certain that the transform methods may yet prove a potent tool for the greater understanding of Z sets.

The Z sets sit within a wider context of polychord content vectors, and set class decompositions in terms of M-classes. Again, Lewin gives a lot of the initial work in this area , though (Collins 1999) demonstrates a Z pair with respect to the trichord vector. Since the embeddings increase in complexity from 2-set embedding (interval classes), we would not expect to solve our interval class problem from this route!

For the current property list for Z sets, we'd have to include many negatives rather than positives:

1. Z sets can be self-transposing
2. Z sets can be self-inverting
3. Z sets do not have unique <1> (spectrums of diatonic length 1)

The positives can be summarised thus

4. Z sets A, B must both be M-classes for some M
5. No Z set representatives are in the P(1,0) relation
6. Under particular conditions of 'dual inversion' given in (Soderberg 1995), Z sets may arise
7. If /A/ is an n/2-class in $\mathbf{Z}_n$, n even, then the complement of A, A', will either give a Z partner /A'/, or /A/ = /A'/

Even though their discovery is often a setback to finding simple properties for Z sets, the Z sets revealed by this paper can be a useful resource. In particular I must mark out the relatively few self transposing Z sets in $\mathbf{Z}_{24}$ as an intriguing chord type for quarter-tonal music. There is a unique pair constructed using orbits of order three, listed in this paper, and 23 examples which can be constructed using the Z pairs in Z_{12} and orbits of size 2. Certainly the transposition invariant Z sets in the quartertone space are special compared to the 46% of set classes with Z partners!

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[1] Lewin's EMB function (Lewin 1987) is avoided in this article as an unnecessary complication.

[2] This could also be expressed as a special product:

$A = \{ 0, 1, 2, 5 \} \oplus \{ 0, 8, 16 \}$

$B = \{ 0, 1, 3, 4 \} \oplus \{ 0, 8, 16 \}$

The plus denotes that the pitch class set A is constructed from any addition of an integer from the left hand set and an integer from the right.

[3] This conclusion is equivalent to /A/ Z /B/ in $\mathbf{Z}_n$ iff $/A_I/$ Z $/B_I/$ in $\mathbf{Z}_{(j,n)}$

[4] The author has not yet verified these values with the Polya theorem derived formulae in (Fripertinger 1999)