# Uniqueness of Pitch Class Spaces, Minimal Bases and Z Partners

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**Abstract**

This paper examines some original aspects of modern music theory associated with measures of uniqueness for pitch class spaces of any size. Certain decompositions shall be defined for the set classes of those spaces based on the interval vector and its generalisations, the M-class vector and polychord content vector. From these, a general Z relationship is developed. It subsumes the Z pairings with respect to Forte's interval vector in the 12 note chromatic pitch class space with inversion and transposition set classes. The results suggest notions of a "minimal basis" for a given pitch class space with respect to a given set class equivalence relation. Uniqueness measures are outlined for these musical spaces.

Finally, similarity relations are introduced for set classes founded on this work and that of many other recent music theorists.

The author seeks to allow understanding without overuse of mathematics; but all proofs are provided. The mathematical approach will follow that of David Lewin's notions of embedding (1987).

## 1 Background

David Lewin (1977) was the first to generalise the notion of Forte's interval vector to vectors representing the decomposition under abstract inclusion of a set class into other set classes. His ideas of embedding are critical herein, and are expounded at length in his book, Generalised Musical Intervals and Transformations (1987).

Measuring set equivalence is another important theme of music theory mathematics. One of the best recent paper on the subject is arguably by Isaacson (1990). Certainly, his ideas are followed in this paper. Morris (1987) provides more background on the topic, and is a worthwhile starting point for modern music theory. Block and Douthett (1994) reveal another thread in this work. However, there are increasing concerns about the use of similarity indexes in analysis (Demske 1995). A recent approach is to introduce saturation measures founded upon fuzzy set theory (Buchler 1997).

Z sets are mentioned in multiple sources. They are a fascinating side effect of the intervallic decomposition of set classes. Set classes, distinct according to inversion and transposition, can still have equal interval vectors. Recently, Stephen Soderberg (1995) has provided mathematical tools for discovering Z pairs or even triples or worse. He credits Lewin with the discovery of Z triples in 1982.

Decompositions into those set classes with representatives of cardinality three have a special place in modern music theory due to the ideas of genera (Forte 1988). Music Analysis Vol 17, No 2 (1998) provides a glimpse of the controversies in this field of analysis. However, the mathematics of the genera constructions are distinct from the decompositions studied within this paper. The genera are historically motivated analysis tools. This paper will regard the set classes as a mathematical subject area.

Musical use follows from the heightened understanding of the underlying mathematics.

# 2 Mathematics

Whilst the work herein stands within the framework of Lewin's Generalised Interval Systems (GIS), the practical examples and commentary will be for the cyclic equal tempered pitch class spaces.

Given GIS (S,IVLS,int), and a group of operations CANON on S, an equivalence relation is naturally implied.

For the pitch class spaces, the group CANON of operations will be the set of all transpositions and inversions, that is, the interval preserving operations. With respect to CANON, the equivalence classes are the standard $T_nI$ set classes (Morris's notation).

## 2.1 Definition COMB(i,j)

This is the number of ways of choosing j elements from i without regard to order. It is the standard combinatorics combination operator:

$$\text{COMB(i,j)}= \quad \frac{j!}{(j-i)!\,i!}$$

## 2.2 Notation E(p)

The cyclic equal tempered pitch class space of order p will be denoted E(p).

E(12) is the standard chromatic pitch space for Western music. Mathematically, E(p) is often considered as the cyclic group Zp.

## 2.3 Notation X, /X/ (after Lewin)

We write X for a (pitch class) set, /X/ for an equivalence class (set class) containing X

/X/ is generated by X under a group CANON of operations. /X/ only makes sense on the understanding of a group CANON; this requirement is henceforth assumed implicitly.

## 2.4 Definition EMB(X,Y) (after Lewin 1987: 105 5.3.1)

Given sets X,Y the embedding number of X in Y, EMB(X,Y) is the number of forms of X (i.e. members of /X/) that are included in Y.

Lewin further defines EMB(/X/,Y), EMB(X,/Y/) and EMB(/X/, /Y/) in the natural way, and deals with the issue of their well-definition. For instance, for EMB(/X/, /Y/): Given any X belonging to /X/ and Y belonging to /Y/, EMB(/X/, /Y/)= EMB(X, Y).

## 2.5 Definition M-class, M-note set class, M-set (M a positive integer)

The synonyms above all refer to a set class whose representatives are of cardinality M.

The cardinality of the set class itself is not the same as the cardinality of its members. The set class cardinality measures the size of the equivalence class.

### 2.6 Definition card(S)

Given any set S, card(S) is the cardinality of the set.

Therefore, by the comments above, card(/X/) is not the same as card(X).

### 2.7 Definition M-class vector (after Lewin 1987: 106 5.3.3)

By the "M-classs vector of Y" we understand the function EMB(/X/,Y) as the variable /X/ runs through the various set classes whose members have cardinality M.

That is, let /X/ run through the M-classes of our GIS. We can begin to compare sets on the basis of their M-class vectors by comparing values of EMB(/X/,Y) for differing Y.

### 2.8 Notation M-vector, M[X] where X is a set

The above synonyms are used as shorthands for the M-class vector

Example The 2-class vector in E(p) is Forte's interval vector. The 3-class vector is the trichord vector mentioned more rarely in the literature. Using Forte's standard list of set classes of E(12), 3[4-1] = [220000000000]. There are 12 entries in this 3-vector to correspond to the 12 possible 3-sets of E(12); those 12 3-class sets are ordered 3-1, 3-2,…,3-12 for the vector. See Appendix B for further information on the ordering of set classes for E(p).

Note. The M-class vector should have a finite number of non-zero entries as long as Y is finite. It will have a finite number of entries if there are only a finite number of M-classes, and this is the case for all E(p). The finiteness property is the origin of the term vector in the above, for the embedding data is an ordered tuple of positive integers.

### 2.9 Definition M-class decomposition, M-decomposition (M a positive integer)

Let the synonyms above refer to the multiset MS of set classes generated by the M-class vector in the following way. For each entry of the M-vector allow that number of copies of the associated embedded set class /X/ into MS.

Note The decomposition (the multiset) is finite where Y was finite.

The decomposition is a little intimidating mathematically, but it can be thought of as deconstructing a set Y into some particular constituent parts. Because of the abstract inclusion embodied by EMB(/X/, Y), we get an abstract decomposition, and a multiset. If we had taken EMB(X,Y) for all X of cardinality M, we would have a more concrete decomposition (just a set of subsets); but a less theoretically interesting one.

We can finally define the Z relation.

### 2.10 Definition Z relation (Forte 1974, reformulated for GIS)

Given sets X and Y, X and Y are in the Z relation iff the 2-class vectors of X and Y are equal. We write X $Z_2$Y.

Forte's Z relation holds between distinct set classes. However, it is recast here in a more general form for sets. A set can be in the Z relation with itself. This is so that the Z relation is an equivalence relation. It partitions the subsets of the musical space into regions sharing the same 2-class vector.

To write the definition for set classes, simply replace X by /X/ and Y by /Y/ in the definitions pre- and post-ceding. The Z relation can be applied as an equivalence relation on the set of set classes by the well-definition of EMB.

The Z relation can be immediately generalised.

### 2.11 Definition Z relation with respect to M-decomposition

Given sets (set classes) X and Y, X and Y are in the Z relation with respect to M-decomposition iff the M-class vectors of X and Y are equal. We write $X\ Z_M Y$.

Again, fixing M, we can generate equivalence classes according to the Z relation with respect to M decomposition, of sets or set classes.

It is profitable to generalise the idea of the M-class vector as well.

### 2.12 Definition C-vector

Let C be an ordered set of equivalence classes of S, where S is the set of members of the GIS. Denote C= {/ci/: i e I} where I is some ordered indexing set. Given set Y, the C-vector of Y is defined as

C[Y] = {EMB(/ci/,Y): i e I}

The C-vector represents the decomposition of a set Y into an arbitrary set of set classes.

Example in E(12) Let C= {2-1, 2-2, 3-1}. Then C[4-1]= [4,3,2].

This generalisation gives birth to the polychord content vector very rarely indicated in the literature (one mention is in Block, Douthett 1994). The way it is presented here is very general because the indexing set is not necessarily finite.

### 2.13 Definition C-decomposition

The multiset derived in the natural way from the C-vector.

To obtain the M-decomposition from the C-decomposition, simply take C as the set of all M-classes.

The Z relation comes along for the ride:

### 2.14 Definition Z relation with respect to C-decomposition.

Given sets X and Y, X and Y are in the Z relation with respect to C-decomposition iff the C-vectors of X and Y are equal. We write $X\ Z_C Y$.

Example. In E(12), let C be the union of all the 2-class sets (the dyads) plus the 3-set 3-1. Then the C-vector has 7 entries. With respect to C decomposition, 4-15 and 4-29 are in the Z relation. For 4-15 $Z_2$ 4-29, and EMB(3-1, 4-15) = EMB(3-1, 4-29) = 0.

### 2.15 Definition ZPARTNERS(C,X)

C a set of sets (set classes) , X a set (set class)
ZPARTNERS(C,X) is the equivalence class of the relation $Z_C$ on the musical space S of a GIS (set of equivalence classes of S), containing X (/X/).

This definition holds separately for sets and set classes, though the case of set classes is naturally inferred. If we deal with set classes, ZPARTNERS(C,/X/) is a set of set classes; if we deal with sets, ZPARTNERS(C,X) is a set of sets.

ZPARTNERS(M,X) is defined analogously for the special case of the M decomposition.

Examples
In E(12) card(ZPARTNERS(2,/X/)) for all /X/ is at most 2. So we only talk of Z pairs.
Soderberg (1995: 98) demonstrates a Z-triple in E(16):
card(ZPARTNERS(2, /{0,1,2,5,8,10}/ ) = 3.

We now have enough of a framework built up to investigate some properties of the sets ZPARTNERS(M,X).

## 3 Nested Z Partners

### 3.1 Proposition

Suppose two sets X,Y have different M-class decompositions, M>=2. Then they have different (M+1)-class decompositions.
Proof. The proposition is proved in Appendix A.

We must assume that the (M+1)-class decomposition can be taken. So card(X) and card (Y) are greater than or equal to M+1.

The logical equivalent is:

### 3.2 Proposition

Suppose two sets have the same (M+1)- decomposition, M>=2. Then they have the same M-decomposition.

By recursion these propositions imply:

### 3.3 Proposition

If the N-decompositions of sets X,Y are the same, then all M-decompositions where 2<=M<=N are the same.

### 3.4 Proposition

If the N–decompositions of sets X,Y are different, then all M-decompositions for M>=N are different, where M does not exceed X or Y’s cardinality.

These propositions are powerful, for they give a nested structure to the sets ZPARTNERS(M,X) for varying M.

### 3.5 Corollary

Given X, 1<M<=card(X), ZPARTNERS(M,X) is a subset of ZPARTNERS(M-1,X).

Proof
If Y belongs to ZPARTNERS(M,X) then the M-decompositions of Y and X are equal. Hence, by the above propositions, the (M-1) decompositions of Y and X are equal, and Y belongs to ZPARTNERS(M,X). QED

The nesting of ZPARTNERS(M,X) sets will form a basis for measuring the uniqueness of a set X with respect to M-decomposition.

For now, let us consider a more practical question. It is an interesting fact of E(12) that Z partners with respect to 2-decomposition exist. But all 3-decompositions are unique. Are there any grounds for the following conjecture?

### 3.6 Conjecture

Let E(p) be an equal tempered pitch class space, p>3. Then the M-decompositions (into $T_nI$ set classes) for p>=M>=3 are unique.

It turns out that there are Z multiples for 3-decompositions. The first example revealed by computer assisted search is in E(18).

**Conjecture Counterexample:**
set class 1 representative : 0 1 4 5 6 7 8 10 12
set class 2 representative: 0 1 2 3 5 6 7 9 13
These set classes share the 3 vector

[355 354 562 544 342 212 221 241 241]

there being 27 3-sets in E(18). The ordering of the 3-sets is that determined by the computer search. See Appendix B for details of the computer search program.

It is expected that there is no M for which an M-decomposition uniqueness theorem holds across all E(p). Computer searches have been implemented up to and including E(24), but 4-set decomposition uniqueness has yet to be broached.

If we wished to search out further counterexamples, in practical terms, the propositions are very applicable. Suppose we seek a 6-decomposition Z pair. Then the pc sets solving this will also be exhibits of 5, 4,3 and 2 decomposition Z pairs.

Appendix C briefly discusses the failure to prove this conjecture.

There is a much weaker uniqueness result that can be proved, though not for general GIS.

### 3.7 Proposition

Let the GIS be E(p), p>=3, with the $T_nI$ set classes. Given set X such that
3 <=card(X), ZPARTNERS(card(X)-1,/X/)={/X/}.
Proof Appendix A

In the sequel, this proposition gives an upper bound on the uniqueness measures for E(p). It guarantees that the chain of nested subsets ZPARTNERS(M, /X/) will become a trivial set before ZPARTNERS(card(X), /X/). It is an open question whether this bound can be improved.

# 4 Uniqueness Measures on Sets, Spaces and Minimal Bases

This section includes some measures of uniqueness, given practical application for the spaces E(p). However, these are not the only measures of worth that might be constructed from the material so far.

### 4.1 Definition Uniqueness of sets

Given set X find the least positive integer U such that
card(ZPARTNERS(U, X))= card(/X/)

This can be specialised for set classes

### 4.2 Definition Uniqueness of set classes

Given set class /X/ find the least positive integer U such that
card(ZPARTNERS(U, /X/))= 1

### 4.3 Proposition

Given set X, the uniqueness of X as a set matches the uniqueness of its set class /X/.

Proof
card(ZPARTNERS(U, /X/))= 1 iff
/X/ is in the $Z_U$ relation only with itself iff
The representative X is in the $Z_U$ relation only with all other X' in /X/ iff
card(ZPARTNERS(U, X)= card(/X/)

QED

We can speak unambiguously of the uniqueness of a set X or set class/X/.

It is now possible to define the uniqueness of a GIS according to the set uniqueness measure.

### 4.4 Definition Uniqueness of a GIS (S,IVLS,int) according to the $Z_M$ relation

Uniqueness U((S,IVLS,int))= supremum { u(X) : X $\subset$ S}where u(X) is the uniqueness of a set X.

By proposition 4.3 U( (S,IVLS,int) )= sup { u( X) : X $\subset$ S}=
sup {u(/X/) :/X/ equivalence class of S under CANON}

Example. The uniqueness of E(12) is 3. The uniqueness of E(18) is 4 (by computer inspection).

The uniqueness of a space tells us the least M such that all M-decompositions of set classes are unique.
It is possible to obtain an even more interesting measure of uniqueness using C-decomposition. We shall construct a minimal collection C of set classes according to some criteria, such that ZPARTNERS(C,/X/) = 1 for all /X/. The collection C will

be termed a minimal basis for the GIS according to the given criteria. The criteria presented here are not the only criteria that might be chosen.

The “criteria” spoken of allows a total ordering relation on the set of collections of set classes.
We now establish one potential total ordering.

### 4.5 Definition MAXORD

Given a collection of set classes C, define
MAXORD(C) = max { M: there is an M-class in C}

### 4.6 Definition MAXORD Ordering

Given collections of set classes C, D , order according to the following three steps, stopping as soon as the order is resolved:

1.wlog If MAXORD(C) <MAXORD(D) C precedes D in the ordering, and we write C < D o/w
2.wlog If card(C) < card(D) C < D o/w
3. If card(C) = card(D) and MAXORD(C) =MAXORD(D) C=D.

We could go further; step 3 could begin an analysis of distribution of the M-classes in C and D. This analysis is not to be undertaken lightly- for example, in E(12), consider collections C= {2-1, 2-2, 3-1, 4-1, 4-2, 5-1} and
D= {2-2, 3-2, 3-3, 3-4, 4-2, 5-2}. Which is to be preferred as a minimum? The question is deferred for this paper.
The reader may wonder why steps 1 and 2 are not the other way around. Would it not be better to have as small a cardinality C as possible as the first determining factor? The reason that MAXORD is most important, is that a C-decomposition cannot be carried out on a set whose cardinality is less than MAXORD(C)[i]. The MAXORD Ordering makes it most important that the collection decomposition be applicable to as many sets as possible.

### 4.7 Definition Minimal Basis

Given a total ordering on the set of collections of set classes in a GIS, define a minimal basis as a minimal collection with respect to that ordering such that ZPARTNERS(C,/X/) = 1 for all /X/ with card(X)>=MAXORD(C)

Example in E(12) According to the MAXORD Ordering, a minimal basis (and the unique one) in E(12) is the collection of the 12 trichords. This can be verified by hand, attempting to extend the collection of all 2-sets by less than the full complement of 3-sets.

## 5 Similarity Measures For C Decomposition

I adapt freely from both Isaacson (1990) and the Block/Douthett paper on scalar products (1994) to define:

### 5.1 Definition WCVSIM (Weighted C-Vector SIMilarity)

Given an ordered collection of set classes C, given set classes /X/,/Y/ (sets X,Y) where card(X),card(Y)>=MAXORD(C)

WCVSIM(/X/, /Y/)= WCVSIM(X,Y)= σ (W.dCV)

where W= $[w_1,\dots,w_k].I_k$, $[w_1,\dots,w_k]$ a weight vector , $I_k$ the k by k identity matrix and dCV is the difference of C-vectors

dCV= C[/X/] –C[/Y/] = C[X] – C[Y]

σ is the standard deviation from statistical mathematics, i.e.:

$$s = \sqrt{\frac{\sum_{i=1}^{k}(z_i - z)^2}{k}}$$

where the $z_i$ are the k components of dCV, and z is their average.

The index k refers to card(C). dCV will have k entries, as does the weighting vector.

## 5.2 Definition CVSIM

If the weighting vector is a string of 1s, so W= $[1,\dots,1].I_{k=} I_k$ then WCVSIM becomes an unweighted version of WCVSIM, indicated as CVSIM.

## 5.3 Definition WMVSIM, MVSIM

Let C be the collection of all M-sets. In this special case, we write WMVSIM as a synonym for WCVSIM and MVSIM as a synonym for CVSIM.

Example 2VSIM is Isaacson’s icSIM.

As in Isaacson’s conclusions about icVSIM, it is noted that CVSIM is a quantitative measure. For qualitative measures adapted to a certain situation, Block and Douthett lead the way with their weighting vectors; when the more general WCVSIM is more useful.

Let C be a collection of set classes such that card(ZPARTNERS(C,/X/)) = 1 for all /X/ with card(X)>=MAXORD(C). Then it is possible to construct a similarity measure to take advantage of the “uniqueness” of each set class with respect to C-decomposition.

We are hoping that the measure will guarantee unique values from any pivot set class to any second set class. However, that hope is naïve, for the statistical measures on k data sets can still find set classes /X/, /Y/ to be an equal distance from pivot /P/ whilst their C-vectors are unique. What we can hope to get from C-vector uniqueness is the isolation of any pivot from its surroundings. No /X/ distinct from the pivot can be zero distance (with respect to the similarity measure) from the pivot.

Actually, we must strengthen the assumptions on the C-vectors to provide such a measure.

## 5.4 Proposition

Let C be a collection of set classes such that ZPARTNERS(C,/X/) = 1 for all /X/ with card(X)>=MAXORD(C). Further, given any /X/,/Y/, dCV= C[/X/] –C[/Y/] is not a scalar multiple of the constant vector [1 … 1] with card(C) entries.

Then CVSIM has the property that CVSIM(/X/,/Y/)=0 iff /X/=/Y/ as long as card(X), card(Y)>=MAXORD(C).

Proof

CVSIM(/X/,/Y/) = $\sigma$ (dCV)= 0 iff
$z_i$-z=0 for all entries $z_i$ in dCV iff
$z_i$=z for all entries $z_i$ in dCV iff
dCV={z … z} (card(C) entries) QED

It is an open question whether the set of hypotheses can be reduced, and in particular, whether the first clause of the proposition implies the second in all cases; for all Generalised Interval Systems and collections C with the first property.

Example 3VSIM in E(12) meets the hypotheses of the proposition. (the first hypothesis has already been dealt with; the second follows by inspection though some cases can be discounted by cardinality arguments). Furthermore, the 3-sets have already been shown to be the minimal basis of E(12) with respect to some sensible criteria. So 3VSIM is a propitious measure on E(12). Its only restriction is it cannot be applied to the 2-sets. It might be suggested it has great potential application to the theory of genera, which relies so heavily on 3-set embedding.

To conclude this section, a viable tactic to find a powerful measure on a GIS is to find a minimal basis C with respect to careful criteria. If this basis fits the second hypothesis of the proposition, the measure CVSIM so generated is guaranteed distinguished properties.

# 6 A note about saturation vectors

The latest fashion in the field of similarity measure on sets is to consider the embedding numbers relative to the bounds of the values that might be taken by those numbers. For example, we might define

## 6.1 Definition sataggC-vector

Let C be an ordered set of equivalence classes of S, where S is the set of members of the GIS. C= {/ci/: i e I} where I is some ordered indexing set. Given set Y, the sataggC-vector of Y is defined as

sataggC[Y] = {EMB(/ci/,Y)/EMB(/ci/, /S/): i e I}

where /S/ is the aggregate set class, generated by the set of members of the GIS. We assume finiteness.

Other types of saturation vector can be defined: Buchler (1997) provides one with respect to the cardinality of the set Y. This saturation vector can still be examined between sets of different cardinality, though its derivation differs in each case.

The main point is that the work of this paper can still be carried out with saturation vectors.

## **Appendix A** *Proofs*

**Proposition 3.1**
Suppose two sets X, Y have different M-class decompositions, M>=2. Then they have different (M+1)-class decompositions.

The case M=1 is excluded because if card(X)=card(Y), the 1-decompositions of X and Y are often the same. A 1-decomposition is usually determined by cardinality alone. This is certainly the case for the $T_nI$ set classes.

We recall a result of Lewin (1987: 107 5.3.5.2)

**Theorem** Given positive integers L<M<N. Let X be a set of cardinality L, Z a set of cardinality N. Then

$$EMB(X,Z) = \frac{\sum EMB(X,/Y/)EMB(/Y/,Z)}{COMB(N-M,N-L)}$$

Where the summation is over all N-classes /Y/.

**Proof**

If X and Y are in the same set class, EMB(Z,X)= EMB(Z,Y) for all lower cardinality Z. In particular, X and Y have the same M-decomposition, so this case is not consistent with the proposition and can be discarded.

If card(X)=card(Y)=(M+1) the proposition is immediate as long as X and Y are not in the same set class, the case already disposed.

If card(X) is not equal to card(Y) consider the cardinality of the (M+1) decompositions. We are assuming X and Y are finite. The M+1 decomposition of X has COMB(M+1,X) members, that of Y COMB(M+1,Y). By cardinality alone, the decompositions differ.

Otherwise, card(X)=card(Y)>(M+1). We are in a position to utilise Lewin’s theorem. Compare the M-class vectors of X and Y. There exists some entry that differs, say for M-class C. Now M< M+1 < card(X). Suppose the (M+1) decompositions of X and Y are the same. Apply Lewin’s theorems to both X and Y, summing over the M+1 classes (denoted /D/) :

$$EMB(C,X) = \frac{\sum EMB(C,/D/)EMB(/D/,X)}{COMB(card(X)-M+1, card(X)-M)} = EMB(C,Y)$$

This follows because the denominator is a constant (card(X)=card(Y)) and throughout the sum, EMB(/D/,X)= EMB(/D/,Y) by the assumption that the (M+1) decompositions are the same. EMB(C, /D/) does not depend on X and Y.

We have contradicted the assumption that EMB(C,X) differs from EMB(C,Y). Hence, the (M+1) decompositions of X and y must differ. QED

**Proposition 3.7**
Let the GIS be E(p), p>=3, with the $T_nI$ set classes. Given X such that 3 <=card(X), card(ZPARTNERS(card(X)-1,/X/))=1.

For the second result proved in this appendix we require the notion of a successive difference set (See Soderberg(1995), under CORD (0.6) or Chrisman (1977)).

**Definition** Successive difference set, difference set, compact circular ordering
In E(p),given finite set N-set X= {a1,…,aN}where the ai are in increasing order, define an ordered N-tuple

CORD(X)= [[ a2-a1, …, a(N+1)-aN ]] where a(N+1)= a1+p.

**Proof**
The (card(X)-1)-decomposition of a set X has card(X) entries, for COMB(card(X)-1,card(X)) = card(X).

Case 1 card(X)=3
The 2 decomposition of a 3-set has 3 members, each an interval. It is always possible to express one interval i or p-i as the sum of the other two. Call those other two a and b; they are sufficient to characterise the 3-set. Take difference set [[a b (p-(a+b)]]. Because the set classes are equivalence classes under transposition *and* inversion, this difference set represents all possibilities.

Case 2 card(X)>=4

Consider two N-sets X,Y such that /Y/ belongs to ZPARTNERS(card(X)-1,/X/). Hence X and Y have the same (N-1) decomposition. Write X and Y as difference sets

CORD(X)= [[x1, … ,xN]]     CORD(Y)= [[y1, … ,yN]]

The N members of the (N-1)- decomposition of X can be written as difference sets thus:

$[[x_1+x_2,x_3,...,x_N]], [[x_1,x_2+x_3,...,x_N]], ... , [[x_1...x_{N-2},x_{N-1}+x_N]], [[x_N+x_1,x_2,...,x_{N-1}]]$ (*)

Note that the largest entry appearing in this decomposition must be a sum of two $x_i$ (if it was a single $x_i$, note that $x_i+x_{(i+1 \bmod p)}$ is a member of some difference set in the (N-1)-decomposition- a contradiction, since all differences are positive).

Y has the same decomposition (*), but in a similar way, using Y as a difference set, we can generate the decomposition for Y explicitly, in terms of the $y_i$ . As for the $x_i$, the largest entry in the decomposition (*) must be a sum of two $y_i$. Without loss of generality (cyclically permute and invert (retrograde order) the difference set, renumbering terms) , take it as $y_1+y_2$. Now match (any) difference set containing $y_1+y_2$ in (*) to $[y_1+y_2,y_3,...y_n]$. This will force n-2 elements $y_i$ to have the same values and order as n-2 $x_i$. All that is left to do is to determine how to partition $y_1+y_2$ to get $y_1$ and $y_2$. There are two cases:

1. All the difference sets in (*) are the same. Then all the $x_i$ are equal (only possible way to generate an (N-1)-decomposition with only one set involved N times) All the $y_i$ must also be equal for the same reason. Since at least two $y_i$ are known (N>=4) the whole set is known. Since these $y_i$ were the same as entries $x_i$, Y=X.

2. There is a distinct difference set in (*) from [$y_1+y_2,y_3,...y_n$]. If we match this to an equivalent in Y's decomposition, the difference set will involve a different sum $y_k + y_{k+1}$. So one of $y_1$ and $y_2$ must appear 'raw' in the difference set. We can use the values we have already obtained to spot at least one raw (unsummed) entry $y_j$, and line up the difference sets of $x_i$ and $y_i$. The $x_i$ and $y_i$ must appear in their indexed order (use inversion if necessary).This pattern match will determine any one of $y_1$, $y_2$- the other will follow immediately by the property of a difference set that the entries sum to p. Since the second difference set was taken from (*) $y_1$ and $y_2$ (directly and as dependent variable) have been expressed as the remaining entries $x_i$ unclaimed before (and in the correct order!) Y=X.

QED

The definition of the difference set and the identification of the largest entry in the proof require properties specific to E(p). There is no chance of generalising the proof to general GIS. The proof will not even work for $T_n$ set classes, since inversion is required!

**Appendix B** *Computer Searching and Prime Forms*

To investigate computationally, the author has written custom software. Given positive integer p, this software is capable of finding all $T_nI$ set classes of E(p), and can determine and compare N-vectors of E(p). The question covered here is that of choosing an ordering for the set classes generated by the program. Forte may have provided a list for E(12), but is there a sensible way to proceed in general?

Let us use a form of binary encoding: given set X={x1,x2,…xN}in E(p), encode this set as a positive integer using the function ENCODE defined thus

$$ENCODE\ (X) = \sum_{i=1}^{N} 2^{x_i}$$

ENCODE is 1-1 onto $Z_2^p$ (proof omitted but trivial). The inverse function $ENCODE^{-1}$ exists (on $Z_2^p$ ) to convert in the opposite direction.

Now use the natural ordering on the integers to order any two sets via ENCODE.

Given sets X,Y $\quad$ X $<_{ENCODE}$ Y iff ENCODE(X) <ENCODE(Y)

Furthermore, we can define a prime form as

PRIMEFORM(X)= $ENCODE^{-1}$ min {ENCODE(X): X belongs to /X/}

This is subtly different from Forte's prime form, which uses the mapping FORTE:

$$FORTE\ (X) = \sum_{i=1}^{N} 2^{x_N - (x_i - x_1)}$$

FORTE is still 1-1 onto $Z_2^p$ (proof omitted, but straightforward).We use it exactly as for ENCODE. Yet it is less intuitive computationally.

To conclude, the listings of N sets for p other than 12 use ENCODE for finding prime forms and for then ordering the associated set classes. In this paper, Forte's ordering has been used for E(12) where the reader will be likely to be familiar with FORTE.

**Appendix C** *Attempts to prove a conjecture*

The author made many and varied attempts to prove the conjecture before finding the counterexample. For E(p) the failure of an inductive proof using 3.7 for the inductive step may be traced to the following false proposition:

False Proposition Given a valid 3-decomposition for an N-set X, let Y be an M-set where 3<=M<N, such that the 3-decomposition of Y is a subset of the 3-decomposition of X. Then Y is an abstract subset of X.

A counterexample is given by 6-3 which has 3-vector [221 011 111 000] and 4-15 which has 3 vector [001 001 110 000]. 4-15 is not an abstract subset of 6-3.

The following paragraph is an informal summary of some attempts by the author to prove the conjecture (couched in terms of E(p))

The main observation that inspired the conjecture in the first place was that the 3-sets can symbolise a "non-trivial permutation" whereas the 2-sets are "trivial permutations". The use of the term permutation refers to the successive difference representation of a pc set, formally set out in Appendix A. Solving a decomposition backwards requires the "fitting together" of a group of lower cardinality pc sets into a larger cardinality one. This can be visualised by considering the form of subsets of a pc set with respect to successive difference representation. A form of "pattern matching" ensues. The failure of the conjecture to be true indicates that there are multiple solutions, multiple "fits" for the set of successive difference patterns, even when those patterns are in some sense non-trivial permutations. Many efforts were made to try and represent the problem in terms of linear equations, to isolate cases in which the theorem fails. The linear equations are between the successive difference representations of the pc set representatives of the distinct set classes sharing the same N-decomposition. However, solid results have been so far elusive.

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[i] There is a way around this that may lead to interesting results. If one wants EMB(X,Y) and card(X) > card (Y), define EMB(X,Y)= EMB (Y,X). However, the trick causes problems for the comparison of C-vectors for X and Y where /X/ is not equal to /Y/, and card(X), card(Y) < MAXORD(C). For then, suppose Z is a MAXORD(C) set in C. EMB(Z,X)= EMB(X,Z) is not comparable to EMB(Z,Y)= EMB(Y,Z).