An investigation into the existence of Z chords

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Abstract: - This paper considers the role of Z related sets in modern music theory mathematics, and seeks a deeper explanation for their existence. To this end, a generalised interval function is introduced based on a metric on the space Z_p. A non-modulo interval function is used to give an existence proof across all pitch class spaces for the Z sets. The relation of this interval function to the standard interval function allows greater understanding of why Z sets can occur. Statistics are examined for the frequency of Z sets in smaller pitch class spaces, and a conjecture is made on the necessity of Z sets for interval functions.

Key-Words: - Pitch Class Space, Set Class, Z Related Chord

1 Introduction

Z related chords, that is, pitch class sets sharing a common interval class vector, have fascinated music theorists and composers since the time of Schoenberg, which would be precisely when music theory mathematics began its rise in composition and analysis. For a resume of their importance, with examples from pieces by Webern and Bartok, see [2]. In as much as the residue class mathematics of pitch can be said to relate to human perception, Z related pitch class sets should be heard as having the same interval decomposition, the same constituent relation between pairs of notes of a given chord. Since Z paired chords cannot be transformed into one another by the interval preserving operations of transposition and inversion, they have formed a tantalising enigma to composers and analysts. In this paper we explore the mathematics of their existence, seeking to reduce the enigma surrounding them.

2 Metrics and interval vectors

2.1 Basic Definitions

The notation of current music theorists is not standardised. A particularly troublesome topic is the definition of the mathematical domain of the pitch class space of any size[1]. To resolve ambiguities, I provide another definition that is clear of mathematics and should make sense to all parties.

2.1.1 (Definition) E(p)

The equal tempered pitch class space of order p with equivalence classes constructed under transposition and inversion will be denoted E(p).

In E(12) we will utilise the set class names of Forte's standard list[2] from [2].
Throughout the paper, we are only concerned with transposition and inversion set classes.

2.1.2 (Definitions) /A/, A, a, card S, M-class set, COMB(M, N)

Following Lewin [5] we notate a set class by /A/, a representative for a set class A and a pitch class by lower case a. The cardinality of set S is card S. An M-class set is a set class whose representatives are of cardinality M. COMB(M, N) is the number of ways of choosing M objects from N.

When dealing mathematically with pitch classes in E(p) we consider the ring Z_p, the integers modulo p, with the inner compositions of addition and multiplication.

2.2 New definitions

2.2.1 (Definition) Interval function

A metric INTFUNC: Z_p x Z_p to Z_p is called an interval function[3]. It must satisfy the metric requirements ([9]: pp 21):
(MET1) INTFUNC(a, b) = 0 iff a = b
(MET2) INTFUNC(a, b) = INTFUNC(b, a) $\forall$ a, b in Z_p
(MET3) INTFUNC(a, b) + INTFUNC(b, c) $\geq$

INTFUNC(a, c)

The two interval functions critical to this paper are:

2.2.2 SIF (standard interval function) on E(p)

SIF(a,b)= min(p+a-b, b-a) where p>=b>a>=0

The image set of SIF over E(p) corresponds to the dyad set classes of E(p). These values are known as the interval classes.

2.2.3 IFNOMOD (interval function without modulo) on E(p)

IFNOMOD(a,b)= b-a where wlog p>b>=a>=0

There is no modulo p and we are just examining a bare interval in the integers. However, the definition forces the value of IFNOMOD(a, b) within the range 0 to p-1, with IFNOMOD(a, b) = 0 iff a= b. IFNOMOD is onto Z_p.

It is important to note that IFNOMOD has only a domain dependence on p. We could define IFNOMOD on the integers and consider its restriction to any Z_p as the specific IFNOMOD for E(p). SIF requires the use of p in the definition. Later, the consequence will be results available across all E(p) using IFNOMOD.

Even though each SIF is critically dependent on p, I will often refer to SIF as if there is one SIF across all E(p). Whilst such a view is valid for IFNOMOD, it is certainly not technically accurate for SIF. I hope the reader will forgive this inaccuracy, the understanding of which avoids many extra words in what follows.

Now we may define an interval vector for a given interval function. For the definition we avoid Lewin's embedding functions (refer to [5]) and explicitly produce the multiset as an ordered set of pairs.

2.2.4 (Definition) Interval Vector for a given Interval Function INTFUNC on E(p)

Let RANGE = { INTFUNC(a, b) ; a, b $\in Z_p$ }.
RANGE is a subset of the integers modulo p. Arrange RANGE in increasing order (with the standard increasing order for the integers restricted to Z_p). Let I be an indexing set for the ordered set. RANGE(i) is that element of RANGE corresponded to by i E I.
Given pitch class set A, define
PAIRS(A)= { (a, b) $\in$ A x A: a$\leq$b}
INVIMAGE(i)(A) = { (a, b) $\in$ PAIRS(A) : INTFUNC(a, b) = RANGE(i) }

Then the interval vector for INTFUNC on E(p) for set class /A/ is the ordered set
IV(INTFUNC, E(p)) (/A/) =
{ {i, c_i} : i $\in$ I , c_i = card(INVIMAGE(i)(A)),
A the prime form of /A/ }

In general use, the dependence of the interval vector on the interval function (and hence E(p)) is implicit and taken from the context. To avoid confusion, we will write $IV_{INTFUNC}$ for a specific interval function INTFUNC. By studying the definition above we see how we are measuring the multiplicities of the mapping INTFUNC. We examine how many elements of a set A map to a given element of the image RANGE. The definition allows pairs (a, a). We could easily ignore this part of an interval vector as we now show.

2.2.5 Proposition

Given INTFUNC, and set classes /A/ != /B/, if card A $\neq$ card B, IV(A) $\neq$ IV (B)
Proof By cardinality considerations. Let

$$\text{SUM}(\ \text{IV(/A/)}\ ) = \sum_{i\in I} c_i$$

Where the c_i and I are as above.
The SUM(IV(/A/)) construct is sufficient to differentiate interval vectors. If we do not allow reflexive pairs (a,a):
SUM(IV(/A/)) = COMB(2, card A) $\neq$
COMB(2, card B)= SUM (IV(/B/))
If we do
SUM(IV(/A/)) = (card A * card A)/2 $\neq$
(card B * card B)/2 = SUM (IV(/B/))

2.2.6 (Definition) $Z_{INTFUNC}$ relation

For an interval function INTFUNC, /A/ $Z_{INTFUNC}$ /B/ iff /A/ $\neq$ /B/ and $IV_{INTFUNC}$(/A/) = $IV_{INTFUNC}$(/B/). /A/ and /B/ are termed $Z_{INTFUNC}$ pairs or $Z_{INTFUNC}$ partners, and individually may be called $Z_{INTFUNC}$ chords.

This is a general Z relation for interval vectors. It would be an equivalence relation if we dropped the condition that /A/ $\neq$ /B/, but we have no need of Z partner equivalence classes in this paper (see [1]). The use of the term Z partners is to avoid the misconception that set classes are only ever Z related in pairs. For instance Soderberg [8] demonstrates Z related triples, quadruples etc for the standard interval function.

In particular, the Z relation we are used to dealing with is Z_{SIF} (called Z_2 in [1]). In the context of this paper, we must be explicit about which interval function we are dealing with for a Z relation.

One result comparing Z_{SIF} to $Z_{IFNOMOD}$ is required:

2.2.7 Proposition

If /A/ $Z_{IFNOMOD}$ /B/, then /A/ Z_{SIF} /B/
Proof Compare the interval functions.

The converse is false. It is not true that 4-Z15 $Z_{IFNOMOD}$ 4-Z29. It has been shown that the $Z_{IFNOMOD}$ relation is a subset of the Z_{SIF} relation on E(p)[4].

2.3 Are there Z Partners for ZIFNOMOD?

In terms of our general aims, it would be beneficial to find a 'stronger' version of the standard interval function, call it IFSTRONG, that does not allow $Z_{IFSTRONG}$ pairs, for then we could analyse exactly what is causing failure of uniqueness of the Z_{SIF} relation. We would require a proof that for all E(p), IFSTRONG interval vectors are unique across the set classes of a given E(p). Attempts to find a proof are demolished by any counterexample.

So we look for $Z_{IFNOMOD}$ partners. Are there any in E(12)? A computer search reveals none. Are there any at all? Attempting to prove there are not is quite productive in an understanding of what is going on, and is discussed later. For now, we give the answer; we would not be able to prove uniqueness of all set classes across all E(p) with respect to the IFNOMOD interval vector, for there is a $Z_{IFNOMOD}$ pair in E(14):

2.3.1 Counterexample

/A/ , prime form A = { 0,1,4,5,6,7,8,9,11 }
/B/ , prime form B = { 0,1,2,3,4,6,7,8,11 }

$IV_{IFNOMOD}$(/A/) = [96555433211100] = $IV_{IFNOMOD}$(/B/)

/A/ and /B/ could not exist for p<12. With respect to E(12) A and B are both aspects of the same set class 9-4. In E(13) they are in separate set classes but are not prime forms, so we cannot compare interval vectors. For all p>14, the forms of /A/ and /B/ as given above are prime forms and are distinct set classes. Let us sum up what this means:

2.3.2 Proposition

There is a common $Z_{IFNOMOD}$ pair for all E(p) with p>=14.
Proof. /A/ and /B/ above will be shown to be that pair. We need two separate but straightforward to prove results. First, that /A/ and /B/ are distinct set classes for all E(p) p>=14, and their prime forms remain as A and B. Simply examine the Boolean pattern of the prime forms to see this is true. Secondly, that IFNOMOD as an interval function does not change value across all E(p) where there are common pitch classes. It was already observed earlier that IFNOMOD on E(p) as an interval function is a restriction of a metric on the integers to domain Z_p, and has no other dependence on p. The result follows.

2.3.3 Corollary

There is a common Z_{SIF} pair with respect to the standard interval vector for all E(p) with $p \geq 14$.
Proof. The previous proposition gives the pair using proposition 2.2.7 that the $Z_{IFNOMOD}$ relation is a subset of the Z_{SIF}.

There is no other theorem on existence of Z_{SIF} partners over all p to my knowledge. For instance, Babbitt's theorem only shows that they can potentially exist, and then for even p only.
If the reader is wondering about the assertion 'for all p' the gaps are quickly filled in by computational results for p below 14.

Perhaps we proceeded in a roundabout way, but we have proved something about the standard Z chords using a particularly special type of Z chord.

3 Enumeration and discovery of Z chords

We know that Z chords for our standard interval vector exist; that is the reason for this study after all! Unfortunately, showing the existence of Z pairs is often a proof by discovery. There is no enumeration theorem[5] and no theorem to predict the relative distribution of Z sets in the E(p). Table 1 gives a listing of the frequencies of Z_{SIF} chords (set classes which are in the Z_{SIF} relation with some other class) found by computer search.

Table 1 Frequencies of Z_{SIF} sets in E(p), 8 £ p £ 22

p	interval classes	#set classes	#Z set classes	proportion
8	4	30	1	3.3%
9	4	46	0	0%
10	5	78	6	7.69%
11	5	126	0	0%
12	6	224	46	20.5%
13	6	380	12	3.16%
14	7	687	144	20.96%
15	7	1224	160	13.07%
16	8	2250	728	32.36%
17	8	4112	368	8.95%
18	9	7685	2766	35.99%
19	9	14310	1296	9.06%
20	10	27012	10403	38.5%
21	10	50964	9268	18.19%
22	11	96909	32085	33.11%

The number of set classes includes the set class represented by the empty set. Trivially, this set class could never be in the Z relation with another set class.

Table 2 Frequencies of $Z_{IFNOMOD}$ sets in E(p), 14 £ p £ 22

p	interval classes	#set classes	#Z sets	proportion
14	14	687	2	0.29%
15	15	1224	6	0.49%
16	16	2250	16	0.71%
17	17	4112	34	0.83%
18	18	7685	62	0.81%
19	19	14310	116	0.81%
20	20	27012	176	0.65%
21	21	50964	306	0.60%
22	22	96909	530	0.55%

The ZIFNOMOD sets are so much rarer that they could be quite fascinating for composition in the larger pitch class spaces.

The proportions for the even p are always higher, one would assume as a consequence of Babbitt's Hexachord Theorem[6]. Note that the only Z pair in E(8) arises in the case where Babbitt's Hexachord Theorem holds.

There is another side to the distribution of the Z chords, in terms of the distribution by M-class. Computer search allows the comparison for E(22) in Table 3. For M not listed, the number of Z chords is zero. The important points are that the values for SIF exhibit symmetry, whilst those for IFNOMOD are asymmetrical. We see the consequences of Babbitt's theorem in the large number of 11-class Z_{SIF} chords. In contrast, the $Z_{IFNOMOD}$ chords are not a t a maximum for M = 11.

Table 3 Distribution by M-class in E(22)

M-class	SIF	IFNOMOD
5	40	0
6	125	6
7	695	22
8	1185	30
9	2975	162
10	3220	66
11	15605	52
12	3220	134
13	2975	20
14	1185	14
15	695	24
16	125	0
17	40	0

How do we find Z pairs? Soderberg [8] provides a methodology that often provides Z pairs, but cannot exhaustively find them, particularly since it cannot work in E(p) for odd p. Until a property list for Z chords can be developed independent of the interval vector itself, computer searches will be the standard tool, and will turn up plenty of Z pairs- table 1 shows how easy they are to discover.

4 Interval class decomposition as loss of reconstruction information

Let us begin by examining the most famous Z_{SIF} related pair, 4Z-15 and 4Z-29, the Z pair of lowest cardinality in E(12).

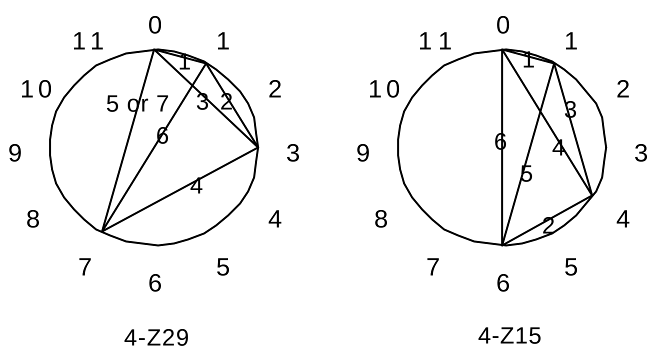

Figure 1

Figure 1 shows Krenek diagrams for the prime forms of these set classes. The chords of the circle indicate the six possible pairings of a cardinality four set (binomial coefficient COMB(2 4)). Integers on the lines reveal the value of the interval functions SIF and IFNOMOD.

There is only one difference between the two functions; that for the chord labelled 5 or 7 in 4-Z29, where the value 5 corresponds to our standard interval function. For SIF looks at the shortest possible distance around the circumference between two points, this being exactly what modulo arithmetic sets up for us. IFNOMOD does not allow a topological short cut, and all the intervals are taken within the span of 0 to 7. This difference of approach is sufficient to differentiate the interval vectors of our set classes with respect to $IV_{IFNOMOD}$, whilst those for IV_{SIF} are equal.

IV_{SIF}(4-Z15)=IV_{SIF}(4-Z29) = [4111111]
$IV_{IFNOMOD}$(4-Z15) = [411111100000]
$IV_{IFNOMOD}$(4-Z29) = [411110110000]

It is the min function that reduces the quality of interval information we have available to the stage where we can no longer differentiate 4-Z15 and 4-Z29.

One realises that order relations are critical and always lurk in the background. Which pairing does each interval come from? We see from the above that for 4-Z15 the interval class 5 comes from one place, but for 4-Z29 another. In general, the information given to us solely by the interval vector does not tell us a unique reconstruction order. In my early (and futile) attempts to prove the uniqueness of IFNOMOD across all p, I discovered that I could always reconstruct the first three elements of a prime form using $IV_{IFNOMOD}$, but thereafter, was unable to choose the next element. There were multiple possible paths to solutions, and the counterexample for E(14) demonstrates how two equally valid reconstruction solutions coexist[7].

To summarise, why could we not prove a theorem about uniqueness of interval vectors with respect to the standard interval function? How do the counterexamples, which are the Z_{SIF} sets, emerge? The answer is in the information contained in the interval vector. We must have sufficient information to reconstruct a single set class, to assert that the interval vectors across all set classes are unique. A theorem of interval vector uniqueness must demonstrate that only one possible set class could give rise to a given interval vector. If such a theorem cannot be proven, then the interval vector as a representation of a set class has less information than the original set class itself.

To provide the denouement, I conjecture on the basis of strong evidence:

4.1.1 Conjecture

For any set of interval functions (metrics) IF: Z_p x Z_p to Z_p defined across all p there is a Z_{IF} chord for some E(p).
or perhaps a better phrasing would be:

4.1.2 Conjecture (rephrased)

Given interval function IF: Z_p x Z_p to Z_p, if $p \geq 14$, there exists at least a single pair of Z_{IF} chords in E(p).
Possible Proof Show an isometry of map to $Z_{IFNOMOD}$. Consequently, a counterexample will occur by E(14). IF must be onto Z_p to better $Z_{IFNOMOD}$'s performance.
Consider E(p) for p=14. The beginning of the proof is straightforward. Consider the values of IF(0, 1), …, IF(0, 13). If IF(0,1) to IF(0,7) are not unique, we immediately get distinct set classes Z related (i.e. {0,1} Z_{IF} {0,2}). The tricky part is now working through all possible metrics, demonstrating further Z_{IF} partners for all cases.

If we allowed an interval function to map to the positive integers instead of Z_p, we may quickly construct exactly what we have been denied so far.

Label the first p primes of the integers
$q_0=2, q_1=3, \ldots, q_p$
IFPRIME(a, b)= $q_a * q_b$ if $a \neq b$, 0 if a=b

Then a given value of IFPRIME encodes the pitch classes from a set form that gave rise to it. Whilst all sets will have a unique IFPRIME vector, so there is no constant vector for a set class, if we only allow interval vectors for the prime forms, we will have obtained uniqueness.

References

[1] Nicholas Collins. "Uniqueness of Pitch Class Spaces, Minimal Bases and Z Partners", *Proceedings of the Diderot Forum on Mathematics and Music (Vienna)* ISBN 3-85403-133-5, 1999, pp 63-77

[2] Allen Forte. *The Structure of Atonal Music*, Yale University Press, 1973

[3] Harald Fripertinger. Enumeration and Construction in Music Theory, *Proceedings of the Diderot Forum on Mathematics and Music (Vienna)* ISBN 3-85403-133-5, 1999, pp 170-203

[4] Harald Fripertinger. Enumeration of Mosaics, *Discrete Mathematics*, No 199, 1999, pp 49-60

[5] David Lewin. *Generalised Musical Intervals and Transformations*, Yale University Press, 1987

[6] Robert D. Morris. *Composition with pitch classes*, Yale University Press, 1987

[7] D.L. Reiner. Enumeration in Music Theory, *Amer. Math. Monthly*, 92, 1985, pp 51-54

[8] Stephen Soderberg. Z-Related Sets as Dual Inversions *Journal of Music Theory*, Vol 39, pp 77-100, 1995

[9] W.A.Sutherland. *Introduction to Metric and Topological Spaces*, Oxford University Press, 1989

[10] Howard J Wilcox. Group Tables and the Generalised Hexachord Theorem, *Perspectives of New Music*, Vol 21, No 2, 1983

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[1] There are great differences based on the mathematical perspective of the researcher. See [3] for a combinatorics viewpoint of Z_n and group D_{12}, [5] for Generalised Interval Systems with S= { 0, …, n-1 } int(a, b) = b-a mod n and CANON the interval preserving operations, or [6] for $[r^{1/n}]$ and T_nI set classes. The standard work for music analysts is [2].

[2] An appendix in [1] discusses the set class names for general E(p).

[3] There are many other mappings we might consider for interval functions, but those that concern us in this paper are metrics. The assertion at the end of the paper requires the interval function to be a metric. There is also one non-standard manipulation; a standard metric as described in [9] is a map into the real numbers, not to Z_p.

[4] The definition of interval vector has one trick within it. It is required that the prime form of a set class be utilised in finding the interval vector. This is because, for IFNOMOD, the interval vector varies across the representatives of the set class! It can be shown that this variation is in a sensible way, but we must avoid any ambiguity over interval vectors if we are to compare them between distinct set classes.
We could change the definitions to eventually define /A/ $Z_{IFNOMOD2}$ /B/ iff there exists form A of /A/ and form B of /B/ such that
$IV_{IFNOMOD}(A)= IV_{IFNOMOD}(B)$.
Is is then easy to show
/A/ $Z_{IFNOMOD2}$ /B/ implies /A/ Z_{SIF} /B/
Yet /A/ Z_{SIF} /B/ implies /A/ $Z_{IFNOMOD2}$ /B/ is false. For a manual check will show 4-Z15 is not in the $Z_{IFNOMOD2}$ relation to 4-Z29. What we are examining here is yet another sub-category of the Z_{SIF} relation. Further, since $Z_{IFNOMOD2}$ subsumes $Z_{IFNOMOD}$, considering relations of the space E(p):
$Z_{IFNOMOD} \subseteq Z_{IFNOMOD2} \subseteq Z_{SIF}$
We use $Z_{IFNOMOD}$ in the text because we must prove that two prime forms are distinct for all $p \geq 14$.

[5] Given the success of the application of Polya's Theorem to musical enumeration ([7], [3], [4]) one wonders whether formulae for the number of Z chords in a given E(p) could be constructed? There is a fundamental difficulty applying Polya's Theorem to enumerate Z chords; the Z relation as a property too complex to easily identify with a group action. It may be that at some future point, headway could be made along this path by finding an appropriate representation susceptible to the theory.

[6] In general, Babbitt's Theorem is a dead end. It is a consequence of complementation, and it does not predict that any Z pairs must occur, only that they will occur where a set classes complement is not itself. For proofs of the theorem see ([5]: pp145, [10]). The Wilcox example is particularly illuminating, but only goes to show how critical the factor of complementation is in the proof. There is no general analogue to Babbitt's Hexachord Theorem for the interval vectors other than SIF, since the proof of the theorem requires the very fact that the range for SIF are the interval classes.

[7] There are a couple of approaches I've found to presenting this as an algebra problem. One is to imagine two possible Z partners and their difference sets (see [1]). The second is to construct a p x p 'delta matrix' using characteristic functions over the elements of Z_p. Neither approach leads to any general way of reducing to a convenient system of linear equations, so these ideas are not presented in the main text.